

# FRACTIONAL LOGARITHMIC SCHRÖDINGER EQUATIONS

PIETRO D'AVENIA, MARCO SQUASSINA, AND MARIANNA ZENARI

**ABSTRACT.** By means of non-smooth critical point theory we obtain existence of infinitely many weak solutions of the fractional Schrödinger equation with logarithmic nonlinearity. We also investigate the Hölder regularity of the weak solutions.

## 1. INTRODUCTION

Let  $s \in (0, 1)$  and  $n > 2s$ . The non-linear fractional logarithmic Schrödinger equation

$$(1.1) \quad i\phi_t - (-\Delta)^s \phi + \phi \log |\phi|^2 = 0 \quad \text{in } \mathbb{R} \times \mathbb{R}^n$$

is a generalization of the classical NLS with logarithmic nonlinearity [7]. For power type nonlinearities the fractional Schrödinger equation was derived by Laskin [16–18] by replacing the Brownian motion in the path integral approach with the so called Lévy flights. Although the equation

$$(1.2) \quad i\phi_t - \Delta \phi + \phi \log |\phi|^2 = 0 \quad \text{in } \mathbb{R} \times \mathbb{R}^n$$

has been ruled out as a fundamental quantum wave equation by very accurate experiments on neutron diffraction, it is currently under discussion if this equation can be adopted as a simplified model for some physical phenomena [1–3, 23]. Its relativistic version, with D'Alembert operator in place of the Laplacian, was first proposed in [20] by Rosen. We refer the reader to [6–8] for existence and uniqueness of solutions of the associated Cauchy problem in a suitable functional framework and to a study of orbital stability, with respect to radial perturbations, of the ground state solution. Although the fractional Laplacian operator  $(-\Delta)^s$ , and more generally pseudodifferential operators, have been a classical topic of functional analysis since long ago, the interest for such operator has constantly increased in the last few years. Nonlocal operators such as  $(-\Delta)^s$  naturally arise in continuum mechanics, phase transition phenomena, population dynamics and game theory, as they are the typical outcome of stochastic stabilization of Lévy processes, see e.g. the work of Caffarelli [4] and the references therein.

In this paper we aim to study the existence of multiple standing waves solutions to (1.1), namely  $\phi(t, x) = e^{i\omega t} u(x)$ , with  $\omega \in \mathbb{R}$ , where  $u \in H^s(\mathbb{R}^n)$  solves the semi-linear elliptic problem

$$(1.3) \quad (-\Delta)^s u + \omega u = u \log u^2 \quad \text{in } \mathbb{R}^n.$$

Without loss of generality we can restrict to  $\omega > 0$ , since if  $u$  is a solution of (1.3) then  $\lambda u$  with  $\lambda \neq 0$  is a solution of  $(-\Delta)^s v + (\omega + \log \lambda^2) v = v \log v^2$ . From a variational point of view, equation (1.3) is formally associated with the functional  $J$  on  $H^s(\mathbb{R}^n)$  defined by

$$J(u) = \frac{1}{2} \int |(-\Delta)^{s/2} u|^2 + \frac{\omega + 1}{2} \int u^2 - \frac{1}{2} \int u^2 \log u^2.$$

The fractional Sobolev space  $H^s(\mathbb{R}^n)$  (see [14]) is continuously embedded in  $L^q(\mathbb{R}^n)$  for all  $2 \leq q \leq 2_s^*$ , where  $2_s^* := 2n/(n - 2s)$  and its closed subspace  $H_{\text{rad}}^s(\mathbb{R}^n)$  is compactly injected in  $L^q(\mathbb{R}^n)$  for  $2 < q < 2_s^*$  (see [19]). Furthermore, by the fractional logarithmic Sobolev inequality (see [9]) we have

$$(1.4) \quad \int u^2 \log \left( \frac{u^2}{\|u\|_2^2} \right) + \left( n + \frac{n}{s} \log a + \log \frac{s\Gamma(\frac{n}{2})}{\Gamma(\frac{n}{2s})} \right) \|u\|_2^2 \leq \frac{a^2}{\pi^s} \|(-\Delta)^{s/2} u\|_2^2, \quad a > 0,$$

2010 *Mathematics Subject Classification.* 34K37, 35Q51, 35Q40.

*Key words and phrases.* Fractional Schrödinger equations, multiplicity of solutions, regularity of solutions.

The first author was partially supported by GNAMPA project *Aspetti differenziali e geometrici nello studio di problemi ellittici quasi-lineari*. The work was partially carried out during a stay of P. d'Avenia at the University of Verona, Italy. He would like to express his gratitude to the Department of Computer Science for the warm hospitality.

for any  $u \in H^s(\mathbb{R}^n)$ . Whence, it is easy to see that  $J$  satisfies this inequality

$$(1.5) \quad J(u) \geq \frac{1}{2} \left[ \left(1 - \frac{a^2}{\pi^s}\right) \|(-\Delta)^{s/2} u\|_2^2 - \|u\|_2^2 \log \|u\|_2^2 + \left(\omega + 1 + n + \frac{n}{s} \log a + \log \frac{s\Gamma(\frac{n}{2})}{\Gamma(\frac{n}{2s})}\right) \|u\|_2^2 \right],$$

for all  $u \in H^s(\mathbb{R}^n)$  and  $a > 0$  small. However, there are elements  $u \in H^s(\mathbb{R}^n)$  such that

$$\int u^2 \log u^2 = -\infty.$$

Thus, in general, the functional fails to be finite as well as of class  $C^1$ . On the other hand, is readily seen that  $J : H^s(\mathbb{R}^n) \rightarrow \mathbb{R} \cup \{+\infty\}$  is lower semi-continuous. For this reasons we use the non-smooth critical point theory developed by Degiovanni and Zani in [12, 13] for suitable classes of lower semi-continuous functionals, which is based on a generalization of the norm of the differential, the weak slope [11]. We say that  $u \in H^s(\mathbb{R}^n)$  is a weak solution to (1.3) if

$$(1.6) \quad \int (-\Delta)^{s/2} u (-\Delta)^{s/2} v + \omega \int uv = \int uv \log u^2, \quad \text{for all } v \in H^s(\mathbb{R}^n) \cap L_c^\infty(\mathbb{R}^n).$$

The main result of the paper is the following.

**Theorem 1.1.** *Problem (1.3) admits a sequence of weak solutions  $(u_k) \subset H_{\text{rad}}^s(\mathbb{R}^n)$  with  $J(u_k) \rightarrow +\infty$ . Furthermore,  $u_k \in C^{0,2s+\sigma}(\mathbb{R}^n)$  for  $s < 1/2$  and  $u_k \in C^{1,2s-1+\sigma}(\mathbb{R}^n)$  for  $s \geq 1/2$ , for some  $\sigma \in (0, 1)$ .*

The result extends to the nonlocal case the results obtained in [10] for the existence of multiple bound states  $(u_k) \subset H_{\text{rad}}^1(\mathbb{R}^n)$  for equation (1.2). Furthermore, it provides Hölder regularity of the solutions depending upon the value of  $s$ , following the strategy outlined in [15]. We point out that, differently from [15], the nonlinearity  $g(t) = t \log t^2$  extended to zero at  $t = 0$  has a very different behaviour at the origin since  $g(t)/t \rightarrow -\infty$  in place of  $g(t)/t \rightarrow 0$  for  $t \rightarrow 0$ , property which also generates, as described above, the loss of smoothness of the functional  $J$  over  $H^s(\mathbb{R}^n)$ . We mention that, in [22], a class of non-autonomous logarithmic Schrödinger equations with 1-periodic potentials was recently investigated and the existence of multiple solutions was obtained by splitting the energy functional into the sum of a  $C^1$  and a convex lower semi-continuous functional and using the critical point theory of [21].

The paper is organized as follows.

In Section 2 we collect some preliminary notions and results.

In Section 3 we prove that the functional satisfies the Palais-Smale condition in the sense specified in [13].

In Section 4 we prove the existence and the Hölder regularity of the radially symmetric weak solutions.

Throughout the proofs the letter  $C$ , unless explicitly stated, will always denote a positive constant whose value may change from line to line. Moreover, the notation  $\int$  will always denote  $\int_{\mathbb{R}^n}$ .

## 2. PRELIMINARY RESULTS

First, for the sake of self-containedness, we recall the definition of fractional Sobolev space and fractional Laplacian. For any  $s \in (0, 1)$  the space  $H^s(\mathbb{R}^n)$  is defined as

$$H^s(\mathbb{R}^n) := \left\{ u \in L^2(\mathbb{R}^n) : \frac{|u(x) - u(y)|}{|x - y|^{n/2+s}} \in L^2(\mathbb{R}^{2n}) \right\}$$

and it is endowed with the norm

$$\|u\| := \left( \int |u|^2 + \int_{\mathbb{R}^{2n}} \frac{|u(x) - u(y)|^2}{|x - y|^{n+2s}} \right)^{1/2}.$$

Let  $\mathcal{S}$  be the Schwartz space of rapidly decaying  $C^\infty$  functions in  $\mathbb{R}^n$ . We have

**Definition 2.1.** *For any  $u \in \mathcal{S}$  and  $s \in (0, 1)$  the fractional Laplacian operator  $(-\Delta)^s$  is defined as*

$$(-\Delta)^s u(x) = -\frac{1}{2} C(n, s) \int \frac{u(x+y) + u(x-y) - 2u(x)}{|y|^{n+2s}},$$

with

$$C(n, s) = \left( \int \frac{1 - \cos \zeta_1}{|\zeta|^{n+2s}} \right)^{-1}.$$

For functions  $u$  with local Hölder continuous derivatives of exponent  $\gamma > 2s - 1$ , the integral defining  $(-\Delta)^s u$  exists finite. Observe that, using [14, Proposition 3.6], for every  $u, v \in H^s(\mathbb{R}^n)$  we have that

$$(2.1) \quad \int (-\Delta)^{s/2} u (-\Delta)^{s/2} v = \frac{C(n, s)}{2} \int_{\mathbb{R}^{2n}} \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{n+2s}}.$$

We now recall some definitions and results of non-smooth critical point theory by Degiovanni and Zani [13] (see also the references therein). Let  $(X, \|\cdot\|_X)$  be a Banach space and  $f : X \rightarrow \mathbb{R}$  be a function. The (critical point) theory we follow is based on a generalized notion of the norm of the derivative, the weak slope. First we defined it for continuous functions and then we extended it for all functions.

**Definition 2.2.** Let  $f : X \rightarrow \mathbb{R}$  be continuous and  $u \in X$ . Then,  $|df|(u)$  is the supremum of the  $\sigma$ 's in  $[0, +\infty)$  such that there exist  $\delta > 0$  and a continuous map  $\mathcal{H} : B_\delta(u) \times [0, \delta] \rightarrow X$ , satisfying

$$d(\mathcal{H}(w, t), w) \leq t, \quad f(\mathcal{H}(w, t)) \leq f(w) - \sigma t,$$

whenever  $w \in B_\delta(u)$  and  $t \in [0, \delta]$ .

Now, we define the function  $\mathcal{G}_f : \text{epi}(f) \mapsto \mathbb{R}$ , where  $\text{epi}(f) := \{(u, \lambda) \in X \times \mathbb{R} \mid f(u) \leq \lambda\}$ , by  $\mathcal{G}_f(u, \lambda) = \lambda$ . If on  $X \times \mathbb{R}$ , we consider the norm  $\|\cdot\|_{X \times \mathbb{R}} = (\|\cdot\|_X^2 + |\cdot|^2)^{1/2}$  and we denote with  $B_\delta(u, \lambda)$  the open ball of center  $(u, \lambda)$  and radius  $\delta > 0$ , we have that the function  $\mathcal{G}_f$  is continuous and Lipschitzian of constant 1 and it allows to generalize the notion of weak slope for non-continuous functions  $f$  as follows (see [5, Proposition 2.3]).

**Proposition 2.3.** For all  $u \in X$  with  $f(u) \in \mathbb{R}$  we have

$$|df|(u) = \begin{cases} \frac{|d\mathcal{G}_f|(u, f(u))}{\sqrt{1 - |d\mathcal{G}_f|(u, f(u))^2}} & \text{if } |d\mathcal{G}_f|(u, f(u)) < 1, \\ +\infty & \text{if } |d\mathcal{G}_f|(u, f(u)) = 1. \end{cases}$$

This equivalent definition allows us to study the continuous function  $\mathcal{G}_f$  instead of the function  $f$ . In some cases it is also useful the notion of equivariant weak slope.

**Definition 2.4.** Let  $f$  be even with  $f(0) \in \mathbb{R}$ . For every  $\lambda \geq f(0)$ , we denote  $|d_{\mathbb{Z}_2} \mathcal{G}_f|(0, \lambda)$  the supremum of the  $\sigma$ 's in  $[0, +\infty[$  such that there exist  $\delta > 0$  and a continuous map  $\mathcal{H} = (\mathcal{H}_1, \mathcal{H}_2) : (B_\delta(0, \lambda) \cap \text{epi}(f)) \times [0, \delta] \rightarrow \text{epi}(f)$ , satisfying

$$\|\mathcal{H}((w, \mu), t) - (w, \mu)\|_{X \times \mathbb{R}} \leq t, \quad \mathcal{H}_2((w, \mu), t) \leq \mu - \sigma t, \quad \mathcal{H}_1((-w, \mu), t) = -\mathcal{H}_1((w, \mu), t),$$

whenever  $(w, \mu) \in B_\delta(0, \lambda) \cap \text{epi}(f)$  and  $t \in [0, \delta]$ .

Then we can give the following

**Definition 2.5.** Let  $c \in \mathbb{R}$ . The function  $f$  satisfies  $(\text{epi})_c$  condition if there exists  $\varepsilon > 0$  such that

$$\inf\{|d\mathcal{G}_f|(u, \lambda) \mid f(u) < \lambda, |\lambda - c| < \varepsilon\} > 0.$$

In this framework we have the following definitions.

**Definition 2.6.**  $u \in X$  is a (lower) critical point of  $f$  if  $f(u) \in \mathbb{R}$  and  $|df|(u) = 0$ .

**Definition 2.7.** Let  $c \in \mathbb{R}$ . A sequence  $\{u_k\} \subset X$  is a Palais-Smale sequence for  $f$  at level  $c$  if  $f(u_k) \rightarrow c$  and  $|df|(u_k) \rightarrow 0$ . Moreover  $f$  satisfies the Palais-Smale condition at level  $c$  if every Palais-Smale sequence for  $f$  at level  $c$  admits a convergent subsequence in  $X$ .

We will apply the following abstract result (see [13, Theorem 2.11]) that is an adaptation of the classical theorem of Ambrosetti-Rabinowitz.

**Theorem 2.8.** Let  $X$  be a Banach space and  $f : X \rightarrow \bar{\mathbb{R}}$  a lower semi-continuous even functional. Assume that  $f(0) = 0$  and there exists a strictly increasing sequence  $\{V_k\}$  of finite-dimensional subspaces of  $X$  with the following properties:

- (1) there exist a closed subspace  $Z$  of  $X$ ,  $\rho > 0$  and  $\alpha > 0$  such that  $X = V_0 \oplus Z$  and for every  $u \in Z$  with  $\|u\|_X = \rho$ ,  $f(u) \geq \alpha$ ;
- (2) there exists a sequence  $\{R_k\} \subset ]\rho, +\infty[$  such that for any  $u \in V_k$  with  $\|u\|_X \geq R_k$ ,  $f(u) \leq 0$ ;
- (3) for every  $c \geq \alpha$ , the function  $f$  satisfies the Palais-Smale condition at level  $c$  and  $(\text{epi})_c$  condition;
- (4)  $|d_{\mathbb{Z}_2} \mathcal{G}_f|(0, \lambda) \neq 0$ , whenever  $\lambda \geq \alpha$ .

Then there exists a sequence  $\{u_k\}$  of critical points of  $f$  such that  $f(u_k) \rightarrow +\infty$ .

Of course, here we need to review some theorems in [13] for the space  $H^s(\mathbb{R}^n)$ . The following result is useful to prove that our functional satisfies the hypothesis of Theorem 2.8. We know that  $H^s(\mathbb{R}^n) \cap L_c^\infty(\mathbb{R}^n)$  is dense in  $H^s(\mathbb{R}^n)$ . Now we prove that every function in  $H^s(\mathbb{R}^n)$  can be seen as the limit of a particular sequence in  $H^s(\mathbb{R}^n) \cap L_c^\infty(\mathbb{R}^n)$ .

**Lemma 2.9.** *For every  $v \in H^s(\mathbb{R}^n)$  there exists a sequence  $\{v_k\}$  in  $H^s(\mathbb{R}^n) \cap L_c^\infty(\mathbb{R}^n)$  strongly convergent to  $v$  in  $H^s(\mathbb{R}^n)$  with  $-v^- \leq v_k \leq v^+$  a.e. in  $\mathbb{R}^n$ .*

*Proof.* Assume first  $v \in H^s(\mathbb{R}^n) \cap L_c^\infty(\mathbb{R}^n)$ . Let  $\vartheta_k : \mathbb{R} \rightarrow [0, 1]$  in  $C^{0,1}$  with Lipschitz constant  $\lambda_k = C/k$ ,  $\text{supt}(\vartheta_k) \subset [-2k, 2k]$ ,  $\vartheta_k(s) = 1$  on  $[-k, k]$ . Let us set  $v_k := \vartheta_k(v)v$ . Then, observe that  $v_k(x) \rightarrow v(x)$  as  $k \rightarrow \infty$  and  $-v^- \leq v_k \leq v^+$  a.e. in  $\mathbb{R}^n$ . We have  $|v_k(x)| \leq |v(x)|$  and

$$\begin{aligned} |v_k(x) - v_k(y)|^2 &= |(\vartheta_k(v(x)) - \vartheta_k(v(y)))v(x) + (v(x) - v(y))\vartheta_k(v(y))|^2 \\ &\leq 2(C|v(x) - v(y)|^2 \|v\|_\infty^2 + |v(x) - v(y)|^2) \leq C|v(x) - v(y)|^2. \end{aligned}$$

Whence  $v_k \in H^s(\mathbb{R}^n) \cap L_c^\infty(\mathbb{R}^n)$  and, by Lebesgue's Theorem  $v_k \rightarrow v$  in  $H^s(\mathbb{R}^n)$ . The general case boils down to the previous case by arguing on  $\max\{\min\{\varphi_j, v^+\}, -v^-\}$  in place of  $v$ , where, by density,  $\varphi_j \in C_c^\infty(\mathbb{R}^n)$  converges strongly to  $v$  in  $H^s(\mathbb{R}^n)$ .  $\square$

**Remark 2.10.** Arguing as in the proof of Lemma 2.9, we can get that, for every  $u \in H_{\text{loc}}^s(\mathbb{R}^n)$ ,

$$H_{\text{loc}}^s(\mathbb{R}^n) := \{u \in L_{\text{loc}}^2(\mathbb{R}^n) : \|(-\Delta)^{s/2} u\|_2 < +\infty\},$$

and  $v \in H^s(\mathbb{R}^n)$ , there exists a sequence  $\{v_k\} \subset V_u$ ,

$$V_u := \{w \in H^s(\mathbb{R}^n) \cap L_c^\infty(\mathbb{R}^n) : u \in L^\infty(\{x \in \mathbb{R}^n : w(x) \neq 0\})\},$$

strongly convergent to  $v$  in  $H^s(\mathbb{R}^n)$  with  $-v^- \leq v_k \leq v^+$  a.e. (see also [12, Theorem 2.3]).

Usually, it is not easy to compute the weak slope of a function. Thus, it is often useful to work with a subdifferential, for which calculus rules hold.

**Definition 2.11.** *For all  $u \in X$  with  $f(u) \in \mathbb{R}$ ,  $v \in X$  and  $\varepsilon > 0$ , we denote by  $f_\varepsilon^0$  the infimum of  $r \in \mathbb{R}$  such that there exists  $\delta > 0$  and a continuous map*

$$\mathcal{V} : (B_\delta(u, f(u)) \cap \text{epi}(f)) \times ]0, \delta] \rightarrow B_\varepsilon(v),$$

such that

$$f(w + t\mathcal{V}((w, \mu t))) \leq \mu + rt,$$

whenever  $(w, \mu) \in B_\delta(u, f(u)) \cap \text{epi}(f)$  and  $t \in ]0, \delta]$ . Then we define

$$f^0(u; v) := \sup_{\varepsilon > 0} f_\varepsilon^0(u, v).$$

As shown in [5, Corollary 4.6], the function  $f^0(u; \cdot)$  is convex, lower semicontinuous and positively homogeneous of degree 1. We can now state the definition of the aforementioned subdifferential.

**Definition 2.12.** *For all  $u \in X$  with  $f(u) \in \mathbb{R}$  we define*

$$\partial f(u) = \{\alpha \in X' : \langle \alpha, v \rangle \leq f^0(u; v), \forall v \in X\}.$$

Now, let us define the continuous functions

$$g(s) := \begin{cases} s \log s^2 & s \neq 0 \\ 0 & s = 0 \end{cases} \quad \text{and} \quad G(s) := \begin{cases} s^2 \log s^2 & s \neq 0 \\ 0 & s = 0 \end{cases}$$

and let

$$(2.2) \quad f(u) := \frac{1}{2} \int G(u) dx.$$

Note that

$$G(s) = 2 \int_0^s (g(t) + t) dt.$$

We have the following preliminary result.

**Proposition 2.13.** *If  $u \in H_{\text{loc}}^s(\mathbb{R}^n)$  we have that:*

- (1) *for every  $v \in H^s(\mathbb{R}^n) \cap L_c^\infty(\mathbb{R}^n)$ ,  $g(u)v \in L^1(\mathbb{R}^n)$ ;*
- (2) *let  $v \in H^s(\mathbb{R}^n)$  and assume that  $(g(u)v)^+ \in L^1(\mathbb{R}^n)$  or  $(g(u)v)^- \in L^1(\mathbb{R}^n)$ , then there exists a sequence  $\{v_k\}$  in  $H^s(\mathbb{R}^n) \cap L_c^\infty(\mathbb{R}^n)$  strongly convergent to  $v$  in  $H^s(\mathbb{R}^n)$  with*

$$\lim_{k \rightarrow \infty} \int g(u)v_k = \int g(u)v.$$

*Proof.* If  $v \in H^s(\mathbb{R}^n) \cap L_c^\infty(\mathbb{R}^n)$ , for  $\delta \in (0, \frac{N+2s}{N-2s})$ , we have

$$\begin{aligned} \int |g(u)v| &\leq \|v\|_\infty \left( \int_{\text{spt}(v) \cap |u| \leq 1} |g(u)| + \int_{\text{spt}(v) \cap |u| > 1} |g(u)| \right) \\ &\leq C \left( 1 + \int_{\text{spt}(v) \cap |u| > 1} |u|^{1+\delta} \right) < +\infty, \end{aligned}$$

then we have (1). To prove (2) we argue as in [12, Theorem 2.7]. Let us assume for instance that  $(g(u)v)^+ \in L^1(\mathbb{R}^n)$  (if  $(g(u)v)^- \in L^1(\mathbb{R}^n)$  the proof is similar). By Lemma 2.9, there is a sequence  $\{v_k\}$  in  $H^s(\mathbb{R}^n) \cap L_c^\infty(\mathbb{R}^n)$  such that  $v_k \rightarrow v$  in  $H^s(\mathbb{R}^n)$  and  $-v^- \leq v_k \leq v^+$  a.e. in  $\mathbb{R}^n$  and, by (1), for every  $k$ ,  $g(u)v_k \in L^1(\mathbb{R}^n)$ . But

$$g(u)v_k = g(u)^+v_k - g(u)^-v_k \leq g(u)^+v^+ + g(u)^-v^- = (g(u)v)^+ \in L^1(\mathbb{R}^n)$$

and by Fatou's Lemma we have

$$\limsup_k \int g(u)v_k \leq \int g(u)v.$$

Hence, if  $\int g(u)v = -\infty$  we conclude, otherwise we have that  $g(u)v \in L^1(\mathbb{R}^n)$  since

$$\int |g(u)v| = \int (g(u)v)^+ + \int (g(u)v)^- = 2 \int (g(u)v)^+ - \int g(u)v,$$

and  $|g(u)v_k| \leq |g(u)v|$ . Thus, by Lebesgue's Theorem we conclude.  $\square$

Moreover we have the following theorem, whose proof is the same of [13, Theorem 3.1].

**Theorem 2.14.** *Let  $u \in H^s(\mathbb{R}^n)$  with  $f(u) \in \mathbb{R}$ . If  $\partial(-f)(u) \neq \emptyset$ , then*

$$\sup \left\{ \int (-g(u) - u)v : v \in H^s(\mathbb{R}^n) \cap L_c^\infty(\mathbb{R}^n), \|v\| \leq 1 \right\} < +\infty,$$

*and hence  $-g(u) - u \in H^{-s}(\mathbb{R}^n)$  upon identification of  $-g(u) - u$  with its unique extension. Furthermore  $\partial(-f)(u) = \{-g(u) - u\}$  and for all  $v \in H^s(\mathbb{R}^n)$  with  $(g(u)v)^+ \in L^1(\mathbb{R}^n)$  or  $(g(u)v)^- \in L^1(\mathbb{R}^n)$ , it holds*

$$\langle -g(u) - u, v \rangle = \int (-g(u) - u)v.$$

*In particular, this holds true for every  $v \in H^s(\mathbb{R}^n) \cap L_c^\infty(\mathbb{R}^n)$ .*

Finally, in our case the  $(\text{epi})_c$  condition and (4) of Theorem 2.8 is easy to prove thanks to the following theorem.

**Theorem 2.15.** *Let  $(u, \lambda) \in \text{epi}(f)$  with  $\lambda > f(u)$ . Then  $|d\mathcal{G}_f|(u, \lambda) = 1$  and, furthermore,  $|d_{\mathbb{Z}_2}\mathcal{G}_f|(0, \lambda) = 1$  for all  $\lambda > f(0)$ .*

The proof can be obtained arguing as in [13, Theorem 3.4].

## 3. PALAIS-SMALE CONDITION

In this section we prove that  $J$  satisfies the Palais-Smale condition, thus we can apply Theorem 2.8 to prove the existence of infinitely many weak solutions to (1.3), namely functions  $u \in H^s(\mathbb{R}^n)$  such that (1.6) holds for any  $v \in H^s(\mathbb{R}^n) \cap L_c^\infty(\mathbb{R}^n)$ . Notice that, that if  $u \in H^s(\mathbb{R}^n)$  and  $v \in H^s(\mathbb{R}^n) \cap L_c^\infty(\mathbb{R}^n)$ , by Proposition 2.13 we can consider

$$(3.1) \quad \langle J'(u), v \rangle = \int (-\Delta)^{s/2} u (-\Delta)^{s/2} v + \omega \int uv - \int uv \log u^2.$$

We will need the following

**Proposition 3.1.** *Let  $u \in H^s(\mathbb{R}^n)$  with  $J(u) \in \mathbb{R}$  and  $|dJ|(u) < +\infty$ . Then the following facts hold:*

(1)  $g(u) \in L_{\text{loc}}^1(\mathbb{R}^n) \cap H^{-s}(\mathbb{R}^n)$  and  $|\langle \alpha_u, v \rangle| \leq |dJ|(u) \|v\|$  for all  $v \in H^s(\mathbb{R}^n)$ , where

$$(3.2) \quad \langle \alpha_u, v \rangle := \int (-\Delta)^{s/2} u (-\Delta)^{s/2} v + (\omega + 1) \int uv + \langle -g(u) - u, v \rangle.$$

In particular, for every  $v \in H^s(\mathbb{R}^n) \cap L_c^\infty(\mathbb{R}^n)$  we have

$$|\langle J'(u), v \rangle| \leq |dJ|(u) \|v\|.$$

(2) if  $v \in H^s(\mathbb{R}^n)$  is such that  $(g(u)v)^+ \in L^1(\mathbb{R}^n)$  or  $(g(u)v)^- \in L^1(\mathbb{R}^n)$ , then  $g(u)v \in L^1(\mathbb{R}^n)$  and identity (3.1) holds.

*Proof.* As in the proof of (1) in Proposition 2.13 we have  $g \in L_{\text{loc}}^1(\mathbb{R}^n)$ . Moreover we can write our functional as  $J(u) = S(u) - f(u)$ , where  $f$  is as in (2.2) and

$$S(u) = \frac{1}{2} \int |(-\Delta)^{s/2} u|^2 + \frac{\omega + 1}{2} \int u^2.$$

Using the properties of the weak slope (see e.g. [5, Theorem 4.13]), we can see that  $\partial J(u) \neq \emptyset$  and, by the calculus rule,  $\partial J(u) \subseteq \partial S(u) + \partial(-f)(u)$  (see [5, Corollary 5.3]), the  $\partial(-f)(u)$  is nonempty too. By Theorem 2.14 we obtain that  $\partial(-f)(u) = \{-u - g(u)\}$ . Since  $S$  is  $C^1$ , again by [5, Corollary 5.3],  $\partial S(u) = \{S'(u)\}$  and then by [5, Theorem 4.13, (iii)], we have  $\partial J(u) = \{\alpha_u\}$  and

$$|dJ|(u) \|v\| \geq \min\{\|\beta\|_{H^{-s}} : \beta \in \partial J(u)\} \|v\| = \|\alpha_u\|_{H^{-s}} \|v\| \geq |\langle \alpha_u, v \rangle|.$$

The second part follows by using (3.2) and assertion (2) of Proposition 2.13.  $\square$

**Remark 3.2.** It is readily seen that  $J$  is lower semi-continuous, see e.g. [10, Proposition 2.2] for the details. Alternatively, one can observe that there exist  $q > 2$  and  $C > 0$  such that  $G(s) \leq C|s|^q$  for all  $s \in \mathbb{R}$ . Then, the assertion follows by a variant of Fatou's lemma.

Finally, we can prove the following

**Proposition 3.3.**  $J|_{H_{\text{rad}}^s(\mathbb{R}^n)}$  satisfies the Palais-Smale condition at level  $c$  for every  $c \in \mathbb{R}$ .

*Proof.* Let  $\{u_k\} \subset H^s(\mathbb{R}^n)$  be a Palais-Smale sequence of  $J$ , i.e.  $J(u_k) \rightarrow c$  and  $|dJ|(u_k) \rightarrow 0$ , thus by Proposition 3.1 we have that  $\langle J'(u_k), v \rangle = o(1) \|v\|$  for any  $v \in H^s(\mathbb{R}^n) \cap L_c^\infty(\mathbb{R}^n)$ . It is easy to see that if  $u \in H^s(\mathbb{R}^n)$ , then  $(u^2 \log u^2)^+ \in L^1(\mathbb{R}^n)$ , thus by (2) Proposition 3.1, the  $u_k$  are admissible test functions in equation (3.1) and

$$(3.3) \quad \|u_k\|_2^2 = 2J(u_k) - \langle J'(u_k), u_k \rangle \leq 2c + o(1) \|u_k\|.$$

Using 1.4, we have that

$$\begin{aligned} \|u_k\|^2 &= 2J(u_k) - \omega \|u_k\|_2^2 + \int u_k^2 \log u_k^2 \\ &\leq 2c + \frac{a^2}{\pi^s} \|(-\Delta)^{s/2} u_k\|_2^2 + \|u_k\|_2^2 \log \|u_k\|_2^2 - \left( \omega + n + \frac{n}{s} \log a + \log \frac{s\Gamma(\frac{n}{2})}{\Gamma(\frac{n}{2s})} \right) \|u_k\|_2^2. \end{aligned}$$

Thus for  $a > 0$  and  $\delta > 0$  small and by (3.3) we have

$$\|u_k\|^2 \leq C + o(1) \|u_k\|^{1+\delta} + o(1) \|u_k\|$$

and so  $\{u_k\}$  is bounded in  $H^s(\mathbb{R}^n)$ . Let  $\{u_k\}$  now be a Palais-Smale sequence for  $J$  in  $H_{\text{rad}}^s(\mathbb{R}^n)$ . By the boundedness of  $\{u_k\}$  and thanks to the compact embedding  $H_{\text{rad}}^s(\mathbb{R}^n) \hookrightarrow L^p(\mathbb{R}^n)$  for  $2 < p < 2_s^*$ , we have that up to a subsequence, there exists  $u \in H_{\text{rad}}^s(\mathbb{R}^n)$  such that

$$u_k \rightharpoonup u \text{ in } H_{\text{rad}}^s(\mathbb{R}^n), \quad u_k \rightarrow u \text{ in } L^p(\mathbb{R}^n), \quad 2 < p < 2_s^*, \quad u_k \rightarrow u \text{ a.e. in } \mathbb{R}^n.$$

We want to prove that for all  $v \in H^s(\mathbb{R}^n) \cap L_c^\infty(\mathbb{R}^n)$

$$(3.4) \quad \int (-\Delta)^{s/2} u (-\Delta)^{s/2} v + \omega \int uv = \int uv \log u^2.$$

So, fixed  $v \in H^s(\mathbb{R}^n) \cap L_c^\infty(\mathbb{R}^n)$ , let us consider  $\vartheta_R(u_k)v$ , where, given  $R > 0$ ,  $\vartheta_R : \mathbb{R} \rightarrow [0, 1]$  is a  $C^{0,1}$  function such that  $\vartheta_R(s) = 1$  for  $|s| \leq R$ ,  $\vartheta_R(s) = 0$  for  $|s| \geq 2R$  and  $|\vartheta_R'(s)| \leq C/R$  in  $\mathbb{R}$ . Obviously, as in Lemma 2.9 we have that  $\vartheta_R(u_k)v \in H^s(\mathbb{R}^n) \cap L_c^\infty(\mathbb{R}^n)$ . Thus, by (3.1) and (2.1) we have

$$\begin{aligned} \langle J'(u_k), \vartheta_R(u_k)v \rangle &= \int (-\Delta)^{s/2} u_k (-\Delta)^{s/2} (\vartheta_R(u_k)v) + \omega \int \vartheta_R(u_k) u_k v - \int \vartheta_R(u_k) u_k v \log u_k^2 \\ &= \frac{C(n, s)}{2} \int_{\mathbb{R}^{2n}} \frac{(u_k(x) - u_k(y))(\vartheta_R(u_k(x))v(x) - \vartheta_R(u_k(y))v(y))}{|x - y|^{n+2s}} \\ &\quad + \omega \int \vartheta_R(u_k) u_k v - \int \vartheta_R(u_k) u_k v \log u_k^2 \\ &= \frac{C(n, s)}{2} \int_{\mathbb{R}^{2n}} \frac{\vartheta_R(u_k(x))(u_k(x) - u_k(y))}{|x - y|^{\frac{n+2s}{2}}} \frac{(v(x) - v(y))}{|x - y|^{\frac{n+2s}{2}}} \\ &\quad + \frac{C(n, s)}{2} \int_{\mathbb{R}^{2n}} \frac{v(y)(\vartheta_R(u_k(x)) - \vartheta_R(u_k(y)))(u_k(x) - u_k(y))}{|x - y|^{n+2s}} \\ &\quad + \omega \int \vartheta_R(u_k) u_k v - \int \vartheta_R(u_k) u_k v \log u_k^2. \end{aligned}$$

Then, we obtain

$$\begin{aligned} &\left| \frac{C(n, s)}{2} \int_{\mathbb{R}^{2n}} \frac{\vartheta_R(u_k(x))(u_k(x) - u_k(y))}{|x - y|^{\frac{n+2s}{2}}} \frac{(v(x) - v(y))}{|x - y|^{\frac{n+2s}{2}}} + \omega \int_{\mathbb{R}^n} \vartheta_R(u_k) u_k v - \int_{\mathbb{R}^n} \vartheta_R(u_k) u_k v \log u_k^2 \right. \\ &\quad \left. - \langle J'(u_k), \vartheta_R(u_k)v \rangle \right| \leq \|v\|_\infty \frac{C}{R} \int_{\mathbb{R}^{2n}} \frac{|u_k(x) - u_k(y)|^2}{|x - y|^{n+2s}} \leq \frac{C}{R}. \end{aligned}$$

Since

$$\vartheta_R(u_k(x)) \frac{(u_k(x) - u_k(y))}{|x - y|^{\frac{n+2s}{2}}} \text{ is bounded in } L^2(\mathbb{R}^{2n}), \quad \frac{(v(x) - v(y))}{|x - y|^{\frac{n+2s}{2}}} \in L^2(\mathbb{R}^{2n}),$$

and

$$\vartheta_R(u_k(x)) \frac{(u_k(x) - u_k(y))}{|x - y|^{\frac{n+2s}{2}}} \rightarrow \vartheta_R(u(x)) \frac{(u(x) - u(y))}{|x - y|^{\frac{n+2s}{2}}} \quad \text{a.e. } (x, y) \in \mathbb{R}^{2n} \text{ as } k \rightarrow +\infty$$

then

$$\int_{\mathbb{R}^{2n}} \frac{(v(x) - v(y))}{|x - y|^{\frac{n+2s}{2}}} \vartheta_R(u_k(x)) \frac{(u_k(x) - u_k(y))}{|x - y|^{\frac{n+2s}{2}}} \rightarrow \int_{\mathbb{R}^{2n}} \frac{\vartheta_R(u(x))(u(x) - u(y))}{|x - y|^{\frac{n+2s}{2}}} \frac{(v(x) - v(y))}{|x - y|^{\frac{n+2s}{2}}}$$

as  $k \rightarrow +\infty$ . In the same way, taking into account that  $\{\vartheta_R(u_k)u_k \log u_k^2\}$  is bounded in  $L_{\text{loc}}^2(\mathbb{R}^n)$  and since  $\vartheta_R(u_k)u_k \log u_k^2 \rightarrow \vartheta_R(u)u \log u^2$  a.e. in  $\mathbb{R}^n$  and, we have

$$\left| \frac{C(n, s)}{2} \int_{\mathbb{R}^{2n}} \frac{\vartheta_R(u(x))(u(x) - u(y))}{|x - y|^{\frac{n+2s}{2}}} \frac{(v(x) - v(y))}{|x - y|^{\frac{n+2s}{2}}} + \omega \int \vartheta_R(u) uv - \int \vartheta_R(u) uv \log u^2 \right| \leq \frac{C}{R}.$$

Thus, letting  $R \rightarrow \infty$ , (2.1) yields (3.4). Moreover, see again Remark 3.2, we have that

$$\limsup_k \int u_k^2 \log u_k^2 \leq \int u^2 \log u^2.$$



Hence, since  $\langle J'(u_k), u_k \rangle \rightarrow 0$  and choosing  $v = u$  in (3.4), we get

$$\limsup_k (\|(-\Delta)^{\frac{s}{2}} u_k\|_2^2 + \omega \|u_k\|_2^2) = \limsup_k \int u_k^2 \log u_k^2 \leq \int u^2 \log u^2 = \|(-\Delta)^{\frac{s}{2}} u\|_2^2 + \omega \|u\|_2^2,$$

which implies the convergence of  $u_k \rightarrow u$  in  $H_{\text{rad}}^s(\mathbb{R}^n)$ .  $\square$

#### 4. PROOF OF THEOREM 1.1

**4.1. Proof for existence.** To prove the existence of sequence  $\{u_k\} \subset H^s(\mathbb{R}^n)$  of weak solutions to (1.3) with  $J(u_k) \rightarrow +\infty$ , we will apply Theorem 2.8 with  $X = H_{\text{rad}}^s(\mathbb{R}^n)$ . By Proposition 3.3 we know that  $J$  satisfies the Palais-Smale condition. Furthermore, by Theorem 2.15 we have that  $J$  satisfies (epi) $_c$  and (4) of Theorem 2.8. Hence, we only have to prove that  $J$  satisfies also the geometrical assumptions. Obviously,  $J(0) = 0$ , and by (1.5),  $J(u) \geq c\|u\|^2$ , for a suitable  $a$  and if  $\|u\|_2$  are sufficiently small. Then, if we take  $Z = H_{\text{rad}}^s(\mathbb{R}^n)$  and  $V_0 = \{0\}$  we have (1). Finally, let  $\{V_k\}$  a strictly increasing sequence of finite-dimensional subspaces of  $H_{\text{rad}}^s(\mathbb{R}^n)$ . Since any norm is equivalent on any  $V_k$ , if  $\{u_m\} \subset V_k$  is such that  $\|u_m\| \rightarrow +\infty$ , then also  $\mu_m := \|u_m\|_2 \rightarrow +\infty$ . Set now  $u_m = \mu_m w_m$ , where  $w_m = \|u_m\|_2^{-1} u_m$ . Thus  $\|w_m\|_2 = 1$ ,  $\|(-\Delta)^{s/2} w_m\|_2 \leq C$  and  $\|w_m\|_\infty \leq C$ , and so

$$J(u_m) = \frac{\mu_m^2}{2} \left( \|(-\Delta)^{s/2} w_m\|_2^2 + \omega + 1 - \log \mu_m^2 - \int w_m^2 \log w_m^2 \right) \leq \frac{\mu_m^2}{2} (C - \log \mu_m^2) \rightarrow -\infty.$$

Thus, there exist  $\{R_k\} \subset ]\rho, +\infty[$  such that for  $u \in V_k$  with  $\|u\| \geq R_k$ ,  $J(u) \leq 0$  and the condition (2) is satisfied.

**4.2. Proof for regularity.** To prove the regularity we follow [15]. First of all we define

$$\mathcal{W}^{\beta,p}(\mathbb{R}^n) = \{u \in L^p(\mathbb{R}^n) : \mathcal{F}^{-1}[(1 + |\xi|^\beta)\hat{u}] \in L^p(\mathbb{R}^n)\}.$$

For the properties of this space, we refer to [15]. Now, let  $u \in H^s(\mathbb{R}^n)$  be a solution of (1.3) and  $\{r_i\}$  a strictly decreasing sequence of positive constants with  $r_0 = 1$ . Let  $B_i = B(0, r_i)$  and define

$$h(x) = u(x) \log u^2(x).$$

We have that

$$(4.1) \quad |h| \leq C_\delta (|u|^{1-\delta} + |u|^{1+\delta})$$

for all  $\delta \in (0, 1)$ . Now let  $\eta_1 \in C^\infty(\mathbb{R}^n)$ ,  $0 \leq \eta_1 \leq 1$ ,  $\eta_1 = 0$  in  $B_0^c$ ,  $\eta_1 = 1$  in  $B_{1/2} := B(0, r_{1/2})$  with  $r_1 < r_{1/2} < r_0$  and  $u_1$  be the solution of

$$(-\Delta)^s u_1 + \omega u_1 = \eta_1 h \quad \text{in } \mathbb{R}^n$$

namely,  $u_1 = \mathcal{K} * (\eta_1 h)$ , where  $\mathcal{K}(x) = \mathcal{F}^{-1}(1/(\omega + |\xi|^{2s}))$  is the Bessel kernel. Then

$$(-\Delta)^s(u - u_1) + \omega(u - u_1) = (1 - \eta_1)h \quad \text{in } \mathbb{R}^n$$

and so

$$u - u_1 = \mathcal{K} * [(1 - \eta_1)h].$$

By Sobolev embedding Theorem,  $u \in L^{q_0}(\mathbb{R}^n)$  with  $q_0 = 2n/(n - 2s)$ . Moreover, by (4.1), [15, Theorem 3.3] and Hölder inequality we have that for a.e.  $x \in B_1$

$$(4.2) \quad |u(x) - u_1(x)| \leq C(\|\mathcal{K}\|_{L^{s_0}(B_{r_{1/2}-r_1}^c)} \|(1 - \eta_1)^{1/(1-\delta)} u\|_{q_0}^{1-\delta} + \|\mathcal{K}\|_{L^{s_1}(B_{r_{1/2}-r_1}^c)} \|(1 - \eta_1)^{1/(1+\delta)} u\|_{q_0}^{1+\delta})$$

where  $s_0 = q_0/(q_0 - 1 + \delta)$ ,  $s_1 = q_0/(q_0 - 1 - \delta)$  and  $\delta < \min\{1, (n + 2s)/(n - 2s)\}$ . In fact,

$$\begin{aligned} |u(x) - u_1(x)| &\leq \int_{B_{1/2}^c} |\mathcal{K}(x - y)| (1 - \eta_1(y)) h(y) dy \\ &\leq C \left( \int_{B_{1/2}^c(x)} |\mathcal{K}|^{s_0} \right)^{1/s_0} \|(1 - \eta_1)^{1/(1-\delta)} u\|_{q_0}^{1-\delta} \\ &\quad + C \left( \int_{B_{1/2}^c(x)} |\mathcal{K}|^{s_1} \right)^{1/s_1} \|(1 - \eta_1)^{1/(1+\delta)} u\|_{q_0}^{1+\delta} \end{aligned}$$



and  $B_{1/2}^c(x) \subset B_{r_{1/2}-r_1}^c$ . Notice that, the same argument shows that for all  $z \in \mathbb{R}^n$  and for a.e.  $x \in B_1(z)$

$$|u(x) - u_1(x)| \leq C(\|\mathcal{K}\|_{L^{s_0}(B_{r_{1/2}-r_1}^c)}\|u\|_{q_0}^{1-\delta} + \|\mathcal{K}\|_{L^{s_1}(B_{r_{1/2}-r_1}^c)}\|u\|_{q_0}^{1+\delta}).$$

and thus, in turn, since the right hand side is independent of the point  $z$ , it follows that  $u - u_1 \in L^\infty(\mathbb{R}^n)$ . Since  $u \in L^{q_0}(\mathbb{R}^n)$  and  $B_0$  is bounded, we have that  $\eta_1 h \in L^{p_1}(\mathbb{R}^n)$  with  $p_1 = q_0/(1 + \delta)$ . Then  $u_1 \in \mathcal{W}^{2s, p_1}(\mathbb{R}^n)$ .

If  $n < 6s$ , then, in (4.1), we take

$$\delta < \min \left\{ 1, \frac{6s - n}{n - 2s} \right\}$$

and so  $p_1 > n/(2s)$ .

If  $n \geq 6s$ , we have that  $p_1 < n/(2s)$  and we proceed as follows. By Sobolev embedding and (4.2) we have that  $u \in L^{q_1}(B_1)$  with  $q_1 = p_1 n / (n - 2s p_1)$ . Then we repeat the procedure, namely we consider  $\eta_2 \in C^\infty(\mathbb{R}^n)$ ,  $0 \leq \eta_2 \leq 1$ ,  $\eta_2 = 0$  in  $B_1^c$ ,  $\eta_2 = 1$  in  $B_{3/2} := B(0, r_{3/2})$  with  $r_2 < r_{3/2} < r_1$ , getting that  $u_2 = \mathcal{K} * (\eta_2 h) \in \mathcal{W}^{2s, p_2}(\mathbb{R}^n)$  with  $p_2 = q_1 / (1 + \delta)$ .

If  $n < 10s$  and in (4.1) we take

$$\delta < \frac{-(n - 4s) + \sqrt{4s(n - s)}}{n - 2s}$$

and we have that  $p_2 > n/(2s)$ .

If  $n \geq 10s$ , then  $p_2 < n/(2s)$  and we iterate this procedure. Straightforward calculations show that

$$(4.3) \quad \frac{1}{q_{j+1}} = \frac{1}{q_1} + \left( \frac{1}{q_1} - \frac{1}{q_0} \right) \sum_{i=1}^j (1 + \delta)^i = \frac{(1 + \delta)^{j+1}}{q_0} - \frac{2s}{n} \sum_{i=0}^j (1 + \delta)^i$$

and, using (4.3), that  $p_j > n/(2s)$  is equivalent to

$$(4.4) \quad (n - 2s)(1 + \delta)^j - 4s \sum_{i=1}^{j-1} (1 + \delta)^i - 4s < 0.$$

From (4.4) we get that, if

$$(4.5) \quad 2(2j - 1)s \leq n < 2(2j + 1)s,$$

then we can take  $\delta$  small enough such that  $p_j > n/(2s)$ . Of course, this procedure stops in  $j$  steps with  $j$  that satisfies (4.5).

Thus, if  $\ell$  is such that  $p_\ell > n/(2s)$ , since  $u_\ell \in \mathcal{W}^{2s, p_\ell}(\mathbb{R}^n)$ , by Sobolev imbeddings (see [15, Theorem 3.2]), we have that  $u_\ell \in C^{0, \mu}(\mathbb{R}^n)$  for  $\mu > 0$  small enough. Moreover, we can estimate  $|u - u_\ell|$  in  $B_\ell$  as in (4.2) and, using the smoothness of  $\mathcal{K}$  away from the origin (see [15, Theorem 3.3]) and since  $|x - y| \geq C > 0$  for  $x \in B_\ell$  and  $y \in B_{\ell-1/2}^c$  we have that for  $x \in B_\ell$

$$|\nabla(u - u_\ell)(x)| \leq \int_{B_{\ell-1/2}^c} |\nabla \mathcal{K}(x - y)| (|u(y)|^{1-\delta} + |u(y)|^{1+\delta}) \leq C(n, s, \|u\|).$$

Then  $u - u_\ell \in W^{1, \infty}(B_\ell)$  and so,  $u - u_\ell \in C^{0, \mu}(B_\ell)$ . Then  $u \in C^{0, \mu}(B_\ell)$  and the  $C^{0, \mu}$ -norm depends on  $n, s, \|u\|_{H^s}$  and on the finite sequence  $r_0, \dots, r_\ell$ . Moving  $B_\ell$  around  $\mathbb{R}^n$  we can recover it, obtaining that  $u \in C^{0, \mu}(\mathbb{R}^n)$  and since in addition  $u \in L^{q_0}(\mathbb{R}^n)$  we get that  $u(x) \rightarrow 0$  as  $|x| \rightarrow +\infty$ .

We now claim that for all  $a, b \in \mathbb{R}$  with  $a < b$  and any  $\delta \in (0, 1)$  we have  $g \in C^{0, 1-\delta}([a, b])$ . Indeed, if  $s, t \in [a, b]$  with e.g.  $t > s > 0$ , then we have

$$|g(t) - g(s)| \leq \int_s^t |g'(\xi)| d\xi \leq 2 \int_s^t (|\log \xi| + 1) d\xi \leq C \int_s^t \xi^{-\delta} = C(t^{1-\delta} - s^{1-\delta}) \leq C(t - s)^{1-\delta}.$$

By symmetry the same inequality holds for negative  $s, t \in [a, b]$ . If  $s, t \in [a, b]$  with e.g.  $s \leq 0 \leq t$ , we get

$$|g(t) - g(s)| \leq |g(t)| + |g(s)| \leq C t^{1-\delta} + C (-s)^{1-\delta} \leq C(t - s)^{1-\delta},$$

proving the claim. Then the regularity assertions of Theorem 1.1 follow by arguing as in [15, p. 1251].

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DIPARTIMENTO DI MECCANICA, MATEMATICA E MANAGEMENT  
 POLITECNICO DI BARI  
 VIA ORABONA 4, I-70125 BARI, ITALY  
 E-mail address: [pietro.davenia@poliba.it](mailto:pietro.davenia@poliba.it)

DIPARTIMENTO DI INFORMATICA  
 UNIVERSITÀ DEGLI STUDI DI VERONA  
 STRADA LE GRAZIE 15, I-37134 VERONA, ITALY  
 E-mail address: [marco.squassina@univr.it](mailto:marco.squassina@univr.it)

DIPARTIMENTO DI MATEMATICA  
 UNIVERSITÀ DEGLI STUDI DI TRENTO  
 VIA SOMMARIVE 14, I-38123 POVO (TN), ITALY  
 E-mail address: [marianna.zenari@univr.it](mailto:marianna.zenari@univr.it)